

Ultra-high energy collisions of nonequatorial geodesic particles near dirty black holes

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We consider collision of two geodesic particles moving around rotating stationary axially symmetric black holes. It is shown for arbitrary nonequatorial motion that under certain conditions the energy in their centre of mass frame can grow unbound (the so-called BSW effect). This generalizes the previous results for equatorial motion around dirty (surrounded by matter) black holes and nonequatorial motion around the Kerr metric. It turns out that the BSW effect occurs near any point of the horizon surface. We do not use special symmetries of space-time typical of the Kerr metric, so the results are quite generic. The general scheme classifying all possible scenarios is discussed.

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I. INTRODUCTION

If two particles moving towards the horizon of a black hole collide, under certain conditions their energy in the centre of mass (CM) frame can become infinitely large. This interesting effect was discovered by Bañados, Silk and West [1] (called the BSW effect after the names of its authors) and is under active study now. It is of interest from the theoretical viewpoint as nontrivial phenomenon in gravity and can have potential astrophysical consequences. Although for the Kerr black hole [2], [3] the products of such collisions have a quite modest energy in the frame of a distant observer due to strong red shift, the observational outcome can become more significant for dirty black holes [4]. In addition, there are hopes on some indirect manifestations of this effect due to new channels of reactions with

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transmutation of particles (in particular, of dark matter) near the black hole horizon [5], [6].

As far as the properties of the BSW effect are concerned, one of the main questions here is to what extent it is universal. In the original paper [1] the extremal Kerr metric was considered and it was assumed that particles move in the equatorial plane. Later on, it was understood that the similar effect reveals itself also for nonextremal black holes [7]. The general picture was described and it was shown that the BSW effect arises for generic dirty (surrounded by matter) rotating black holes due to the properties of the horizon, so it can be viewed as a manifestation of universality of black hole physics [8]. However, this feature was traced for motion in the equatorial plane only thus somewhat restricting the statement about the universality of the BSW effect. Meanwhile, for nonequatorial motion direct collisions between particles near extremal Kerr black hole also lead to the BSW effect but with another kind of restriction: it was found to occur in the bounded belt centered near the equator [9].

The aim of the present paper is to combine both factor and consider the BSW effect for dirty black holes for nonequatorial motion of colliding geodesic particles to generalize previous results [8] and [9]. The new qualitative results consists in that we show that the BSW effect may occur in the vicinity of any point of the horizon and in this sense it retains its universality. There is no contradiction here with the results of [9] since there are different types of the BSW effect. Their classification was discussed in [9] for the Kerr metric and is now extended to dirty black holes in the present work. In particular, the restriction to the belt found in [9] concerns the BSW effect due to direct collisions whereas the BSW effect in polar region requires multiple scattering similar to the BSW effect for nonextremal black holes [7], [8].

The essential point of the analysis in [9] consisted in the fact that the Kerr metric possesses a remarkable property - separation of variables in the Hamilton - Jacobi equation [10]. It does not hold for a generic black hole space-time that does not allow to extend the approach of [9]. Meanwhile, we suggest a more simple general approach that does not demand separability of variables and applies to an arbitrary axially symmetric dirty black hole.

One reservation is order. We do not address in this paper an important issue about the possibility to evade limitations on the energy detected at infinity found earlier for equatorial motion [2] - [4]. However, the present work can serve as a basis for further investigation of this issue.

II. BASIC FORMULAS AND LIMITING TRANSITIONS

Consider the generic axially symmetric metric. It can be written as

$$ds^2 = -N^2 dt^2 + g_{\phi\phi}(d\phi - \omega dt)^2 + \frac{\rho^2}{\Delta} dr^2 + g_{\theta\theta} d\theta^2. \quad (1)$$

Here, the metric coefficients do not depend on t and ϕ . On the horizon $N = 0$. In (1), the factor $\Delta(r) \sim N^2$ is singled out for convenience. The coefficient ρ can depend on θ . Near the horizon, $\Delta \sim N^2$, so r is the analog of the quasiglobal coordinate used in the spherically symmetric case [11]. It is worth noting that the form of the metric somewhat differs from that in the Gauss normal coordinates used in [12], [13], [14]. It is more convenient for our purposes and, in particular, facilitates the comparison to the Kerr metric.

In the space-time under discussion there are two conserved quantities $E \equiv -mu_0$ and $L \equiv mu_\phi$ where $u^\mu = \frac{dx^\mu}{d\tau}$ is the four-velocity of a test particle having the mass m , τ is the proper time and $x^\mu = (t, \phi, r, \theta)$ are coordinates. The aforementioned conserved quantities have the physical meaning of the energy (or frequency for a light-like particle) and azimuthal component of the angular momentum, respectively. Then, using these first integrals for such geodesics one can write down equation of motion (dot denotes the derivative with respect to the proper time τ):

$$m\dot{t} = mu^0 = \frac{X}{N^2}, \quad X = E - \omega L. \quad (2)$$

We assume that $\dot{t} > 0$, so that $E - \omega L \geq 0$.

$$m\dot{\phi} = \frac{L}{g} + \frac{\omega X}{N^2}, \quad g = g_{\phi\phi}. \quad (3)$$

$$\frac{\rho^2}{\Delta} m^2 \dot{r}^2 = V_{eff} \equiv \frac{X^2}{N^2} - \frac{L^2}{g} - m^2 - m^2 g_{\theta\theta} \dot{\theta}^2 \equiv \frac{Z^2}{N^2}. \quad (4)$$

Here, $V_{eff} = \frac{Z^2}{N^2}$ has the meaning of the effective potential.

The quantity which is relevant for us is the energy in the centre of mass frame $E_{c.m.}$ [1] where

$$E_{c.m.}^2 = -(m_1 u_1^\mu + m_2 u_2^\mu)(m_1 u_{1\mu} + m_2 u_{2\mu}), \quad (5)$$

subscript $i=1,2$ enumerates particles. After simple manipulations, one obtains from (2) - (4) that

$$E_{c.m.}^2 = m_1^2 + m_2^2 + 2m_1 m_2 \gamma, \quad \gamma = -u_1^\mu u_{2\mu}, \quad (6)$$

$$\gamma = c - d - g_{\theta\theta} \dot{\theta}_1 \dot{\theta}_2, \quad c = \frac{X_1 X_2 - Z_1 Z_2}{m_1 m_2 N^2}, \quad d = \frac{L_1 L_2}{m_1 m_2 g_{\phi\phi}}. \quad (7)$$

As is known, the BSW effect arises when one of two colliding particle is critical (near-critical) and the other one is usual. By definition, a particle is usual if $X_H \neq 0$ and is critical if $X_H = 0$, $E = \omega_H L$ or near-critical if X_H is small. Here, subscript "H" means that a corresponding quantity is calculated on the horizon. Let particle 1 be critical or near-critical and particle 2 be usual. It follows from (6) that in the near-horizon region where $N \rightarrow 0$, $(Z_2)_H \approx (X_2)_H$,

$$E_{c.m.}^2 \approx 2 \frac{(X_2)_H}{N^2} (X_1 - Z_1). \quad (8)$$

From now on, we consider two cases separately.

III. EXTREMAL BLACK HOLES

A. Infinite growth of energy in the CM frame

Let a particle be not exactly critical but near critical, so

$$L = \frac{E}{\omega_H} (1 - \delta), \quad \delta \ll 1. \quad (9)$$

Then,

$$X \approx E \left(1 - \frac{\omega}{\omega_H} + \delta\right). \quad (10)$$

Near the horizon of the extremal black hole, $\omega - \omega_H$ has the order N [14], so

$$\omega = \omega_H - B(\theta)N + O(N^2). \quad (11)$$

We also adjust δ to have the same order N , so

$$\delta = CN + O(N^2). \quad (12)$$

In particular, one can choose $C = 0$ as it was actually done in [9].

Then, for the near-critical particle we have near the horizon

$$X \approx \frac{E}{\omega_H} \tilde{B} N, \quad \tilde{B} = B + C\omega_H. \quad (13)$$

It is seen that the right hand side of (4) is finite for such a particle,

$$(V_{eff})_H \approx \alpha - m^2 g_{\theta\theta} \dot{\theta}^2, \quad \alpha \equiv \frac{E^2}{\omega_H^2} (\tilde{B}_H^2 - \frac{1}{g_H}) - m^2. \quad (14)$$

Then, it follows from (8) that

$$E_{c.m.}^2 \approx 2 \frac{(X_2)_H}{N} \beta. \quad (15)$$

$$\beta = \frac{E_1 \tilde{B}}{\omega_H} - \sqrt{\alpha_1 - m_1^2 g_{\theta\theta} \dot{\theta}_1^2}, \quad (16)$$

it is assumed that particle 1 is near-critical, particle 2 is usual.

In the limit $N \rightarrow 0$ the CM energy grows unbound: $E_{c.m.} \sim N^{-1/2}$ similar to the equatorial motion case [8], so the BSW effect takes place.

B. Kinematic conditions

This is not the end of story since for the realization of the BSW it is necessary that a critical particle approach the horizon. This gives rise to the condition $Z_1^2 \geq 0$ that entails

$$\alpha_1 \geq 0, \quad (17)$$

whence

$$C \geq \eta(\theta) \equiv \sqrt{\frac{1}{\omega_H^2 g_H} + \frac{m_1^2}{E_1^2} - \frac{B}{\omega_H}}. \quad (18)$$

The value of θ where $\alpha_1 = 0$ (which is equivalent to $C = \eta(\theta)$) is just the turning point for a variable θ . In the case of equatorial motion $\theta = \frac{\pi}{2} = \text{const}$ the term $m_1^2 g_{\theta\theta} \dot{\theta}_1^2$ in (14) is identically zero. If $\eta \leq 0$ in some interval of θ , one can simply put there $C = 0$. The corresponding region generalizes the belt obtained in [9] for critical particles. Inside this region, direct collisions lead to the BSW effect since particle 1 is exactly critical. Outside this belt, we have $\eta > 0$ and the critical particle ($\delta = C = 0$) cannot reach the horizon since condition (17) is violated in its vicinity. In the region $\eta > 0$, a particle should be near-critical (not exactly critical) with C satisfying eq. (18). Here, the quantities g_H and B depend, in general, on θ . Therefore, if one wants to arrange collision between particles near the point on the horizon with some value θ^* of the polar angle, the corresponding minimum value of C also depends on θ^* according to (18). If $m \ll E$, condition (18) simplifies to $C\omega_H \geq \frac{1}{\sqrt{g_H}} - B$.

Near the polar axis, in the absence of the conical defect $g \sim \theta^2$, so in the limit $\theta \rightarrow 0$, we obtain that the admissible C grows like $\frac{1}{\theta}$. To be consistent with the condition (9), collision should occur so closely to the horizon that $N \ll A\theta$ where A is a constant. Then, it follows from (14) - (16) that $E_{c.m.}^2$ is proportional to δ^{-1} .

To realize the BSW effect in the region forbidden for the critical particle near the horizon, the scenario of multiple scattering [7] can be used. A usual particle can approach the horizon and, near it, get a near-critical value of the angular momentum in some collision and only after that collide one more time with a usual particle thus producing the BSW effect.

IV. EXTREMAL KERR METRIC

Let us consider the Kerr metric. In the Boyer-Lindquist coordinates [15], it can be written as

$$ds^2 = -dt^2 \left(1 - \frac{2Mr}{\rho^2}\right) - \frac{4Mar \sin^2 \theta}{\rho^2} d\phi dt + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2 \sin^2 \theta}{\rho^2}\right) \sin^2 \theta d\phi^2, \quad (19)$$

where $\rho^2 = r^2 + a^2 \cos^2 \theta$, $\Delta = r^2 - 2Mr + a^2$, M is the black hole mass, a characterizes its angular momentum. It follows from (19) that

$$\omega = \frac{2aMr}{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}, \quad (20)$$

$$N^2 = \frac{\Delta \rho^2}{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}. \quad (21)$$

For the extremal horizon, $M = a$, and comparing the exact expressions with the near-horizon expansion (11) one can find easily that

$$\omega_H = \frac{1}{2M}, \quad B = \frac{1}{M\sqrt{1 + \cos^2 \theta}}, \quad g_H = 4 \frac{M^2 \sin^2 \theta}{1 + \cos^2 \theta}. \quad (22)$$

Then, (18) gives us

$$C\sqrt{1 + \cos^2 \theta} \geq \sqrt{\frac{(1 + \cos^2 \theta)^2}{\sin^2 \theta} + \frac{m_1^2(1 + \cos^2 \theta)}{E_1^2}} - 2. \quad (23)$$

If $C = 0$, (23) takes the form

$$(m_1^2 - E_1^2) \sin^4 \theta + 2(4E_1^2 - m_1^2) \sin^2 \theta - 4E_1^2 \geq 0 \quad (24)$$

that coincides exactly with eq. (4.5) of Ref. [9], so further analysis developed in [9] applies. The appearance of the belt restricting the region of the BSW effect follows just from (24). However, if one adjusts $C \neq 0$ that satisfies (23), this effect can occur near any polar angle including the region forbidden for pure critical particles.

V. NONEXTREMAL BLACK HOLES

For the nonextremal horizon, in the metric coefficient ω the correction to its horizon value ω_H has the order N^2 [13], [14]. Therefore, for the critical particle $X^2 \sim N^4 \ll N^2$, so in (4) $Z_1^2 < 0$ which means that such a particle cannot reach the horizon in agreement with previous observations [8], [16]. Let a particle be near-critical with δ having the form (12). Then, neglecting in (10) the term of the order N^2 that comes from $\omega - \omega_H$, we obtain

$$X \approx ECN. \quad (25)$$

instead of (13).

The general expression (15) holds with

$$\beta = E_1 C - \sqrt{\alpha - m^2 g_{\theta\theta} \dot{\theta}_i^2}, \quad (26)$$

$$\alpha = E^2 \left(C^2 - \frac{1}{\omega_H^2 g_H} \right) - m^2. \quad (27)$$

The restriction on C takes the form

$$C \geq \sqrt{\frac{m^2}{E^2} + \frac{1}{\omega_H^2 g_H(\theta)}}. \quad (28)$$

The inequality (28) generalizes the restriction derived in eq. (18) of Ref. [8] where it was assumed that $\theta = \frac{\pi}{2}$ (which, in turn, generalized eq. (18) of [7]).

VI. DIFFERENT SCENARIOS OF THE BSW EFFECT

In [9] classification of different scenarios of collisions leading to the BSW effect was suggested. It is based on behavior of the effective radial potential for motion of critical particles in the vicinity of the horizon and takes into account the type of the horizon (extremal or nonextremal). The possibility to single out pure radial motion is based on the fact that variables in the Hamilton - Jacobi equation are separated for the Kerr metric. Now, for generic dirty black holes this is, generally speaking, not so. Nonetheless, it follows from the above consideration that this scheme is extendable to the general case of dirty black holes. In doing so, the relevant quantity that replaces the Boyer-Lindquist coordinate r used in [9] for the Kerr metric is the lapse function N . For nonextremal black holes $N^2 \sim r - r_H$, for extremal ones $N \sim r - r_H$. Then, one obtains the scenario of type I if near the horizon

$Z^2 \approx AN^2$, $A > 0$ (with the extremal horizon), type II if $A = 0$, type III if $A < 0$ (with the extremal horizon), type IV if $A < 0$ with a nonextremal horizon.

Type I corresponds to direct collisions since the potential has the correct sign near the horizon, so a critical particle can safely reach it. The BSW process considered in the pioneering work [1] belongs just to this type. Type II corresponds to circular orbits. The BSW effect due to collisions of particles on such orbits was considered in [17] for the Kerr metric and in [18] for dirty black holes. Type III was mentioned in [9] as the case not discussed in literature before. Meanwhile, in our context, type III is especially interesting since one can recognize here just the BSW effect in polar regions for extremal horizon that is the main subject of the present paper! Type IV is generalization of scenario considered in [7] for the equatorial motion in the Kerr space-time. Formally, one can also consider type V: nonextremal horizons with if $A > 0$. It was mentioned in [9] but not included in the table since it cannot be realized for the Kerr metric. Meanwhile, it is clear from the above consideration that it cannot be realized in general as well. Indeed, the correct sign of Z^2 would mean that the critical particle can reach a nonextremal horizon. This is impossible, as is explained in the previous section.

VII. SUMMARY AND CONCLUSIONS

Thus we considered collision of geodesic particles for arbitrary nonequatorial motion and showed that the BSW effect occurs in the vicinity of generic dirty rotating axially symmetric stationary black holes. This happens not only inside some parts of the horizon surface but near an arbitrary point on it. These results fill some gaps in earlier results and makes the picture complete. All possible scenarios are united in a scheme that generalizes the previous results for the Kerr metric. It turns out that the BSW effect reveals itself irrespective of the presence or absence of special properties of a space-time like separability variables of variables in the Hamilton - Jacobi equation.

Meanwhile, it should be mentioned that our consideration is based on a simplified picture of motion along geodesics. In more realistic circumstances, one should also take into account additional effects like gravitational radiation [19], [20], synchrotron radiation of charged particles in the magnetic field [21], etc. Then, the whole picture can drastically change and the key question is whether these effects can set a limit to the energy that can be reached.

At present, the answer is not obvious since there are indications that the BSW effect retains its validity (at least, for neutral particles) provided there are critical trajectories of general character, even if they are not geodesics [22]. Further careful investigation is needed here.

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